FREE SAMPLE CHAPTER

The Perfect Math Tutor for All

Book 2

Quadratic Functions Properties
And Related Engineering Applications

Chapter 7

What is X? And What is Y?

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Chapter 7
What is X?
And
What is Y?

From now on, as opposed to the First Workbook, we will be dealing exclusively with Applications related to Quadratic Functions.

Created Scenario 1:

Given two numbers X and Y, one number is 500 more than ten times the other number. Given the fact that the Product of the two numbers is a Minimum, what is X? and what is Y?

1. **Let's translate together the given information into equations:**

$$\Rightarrow \begin{cases} Y = 10X + 500 \langle 1 \rangle \\ Minimum: P(X) = X \times Y \langle 2 \rangle \end{cases}$$

- **Once we substitute the value of Y from the equation $\langle 1 \rangle$, the equation $\langle 2 \rangle$ becomes:** $\Rightarrow Minimum: P(X) = X \times (10X + 500) \langle 2 \rangle$
- **After expansion, the Product Function, $P(X)$, becomes a Quadratic Function as follows:** $\Rightarrow Minimum: P(X) = 10X^2 + 500X$

2. **Let's solve the related Quadratic Equation, $10X^2 + 500X = 0$, by factoring:**

$$\Rightarrow Minimum: P(X) = 10X^2 + 500X = 0$$
$$\Rightarrow 10X^2 + 500X = 0$$
$$\Rightarrow 10X(X + 50) = 0$$
$$\Rightarrow \begin{cases} 10X = 0 \\ X + 50 = 0 \end{cases}$$
$$\Rightarrow \begin{cases} X_1 = 0 \\ X_2 = -50 \end{cases}$$

- **For** $X_1 = 0 \Rightarrow Y = 10(0) + 500 = 500$

So, the numbers are (0, 500)

- **For** $X_2 = -50 \Rightarrow Y = 10(-50) + 500 = 0$

So, the numbers are $(-50, 0)$

3. **Let's solve the related Quadratic Equation, $10X^2 + 500X = 0$, by using Quadratic Formula:**

$$\Rightarrow 10X^2 + 500X = 0 \Rightarrow aX^2 + bX + c = 0 \Rightarrow \begin{cases} a = 10 \\ b = 500 \\ c = 0 \end{cases}$$

- **Given $b = 500$ is an Even Number, we apply the discriminant:**

$$\Rightarrow \Delta' = (b')^2 - ac \text{ such as } b' = \frac{b}{2} \Rightarrow \Delta' = \left(\frac{500}{2}\right)^2 - 10(0) = 250^2 = 62500$$

The roots of the equation are:

$$\Rightarrow \begin{cases} \dfrac{-250 - \sqrt{62500}}{10} = \dfrac{-250 - 250}{10} = \dfrac{-500}{10} = -50 \\ \dfrac{-250 + \sqrt{62500}}{10} = \dfrac{-250 + 250}{10} = \dfrac{0}{10} = 0 \end{cases}$$

- **We found the very same roots as we did through factorization process:**

$$\Rightarrow \begin{cases} X_1 = 0 \\ X_2 = -50 \end{cases}$$

- ***We can determine* the Critical Point or the Optimum Value by finding the Mean of the two roots X_1 and X_2** $\Rightarrow Mean = \dfrac{X_1 + X_2}{2} = \dfrac{0 + (-50)}{2} = -25$

- **$X = -25$ is the Critical Value or the Optimum Value or the x-coordinate of the Vertex •**

- **The y-coordinate of the Vertex is easily found by plugging $X = -25$ in $P(X)$**

$$\Rightarrow P(-25) = 10(-25)^2 + 500(-25) \Rightarrow P(-25) = 10(625) + (-12500)$$

$$\Rightarrow P(-25) = 6250 - 12500 \Rightarrow P(-25) = -6250 \Rightarrow Vertex = V(-25, -6250)$$

- **When $X = -25$, $P(X)$ achieves a Minimum of -6250**

4. Let's find the Vertex by completing the square of
$$P(X)=10X^2+500X$$

- Let's factor out 10 as a common factor of both terms $10X^2$ and $500X$

$$\Rightarrow P(X)=10\left(X^2+50X\right)$$

- Let's take the half the coefficient of X , 50, and square it:

$$\Rightarrow 50\to\frac{1}{2}(50)\to 25\to(25)^2\to 625$$

- Let's add and subtract 625 in $P(X)=10\left(X^2+50X\right)$

$$\Rightarrow P(X)=10\left(X^2+50X+625-625\right)\ \langle 1\rangle$$

- We achieved a Perfect Square: $\Rightarrow X^2+50X+625=(X+25)^2$

- Let's plug $(X+25)^2$ in $\langle 1\rangle \Rightarrow P(X)=10\left((X+25)^2-625\right)$

- After distribution, we achieved the Standard Form:

$$\Rightarrow P(X)=10(X+25)^2-6250 = a(X-p)^2+q \text{ Where } \begin{cases} a=10 \\ p=-25 \\ q=-6250 \end{cases}$$

- When $X=-25$, $P(X)$ achieves a Minimum of -6250

$$\Rightarrow Vertex=V(-25,-6250)$$

- $X=-25$ is the Critical Value or the Optimum Value

5. **Let's find the Vertex by applying de derivative of $P(X)$:**

- **Given:** $P(X)=10X^2+500X$

$$\Rightarrow \frac{d}{dx}P(X)=2(10)X^{2-1}+1(500)X^{1-1}$$
$$\Rightarrow \frac{d}{dx}P(X)=20X^1+500X^0$$
$$\Rightarrow \frac{d}{dx}P(X)=20X+500(1)$$
$$\Rightarrow \frac{d}{dx}P(X)=20X+500$$

- **To find the Critical Point, we set the derivative to zero:**

$$P'(X)=\frac{d}{dx}P(X)=0$$
$$\Rightarrow 20X+500=0$$
$$\Rightarrow 20X=-500$$
$$\Rightarrow \frac{20X}{20}=\frac{-500}{20}$$
$$\Rightarrow X=-25$$

- **We finally found the very same x-coordinate of the Vertex or the Critical Point or the Optimum Value as we did by using different methods.**

6. Confidence Interval:

a) Let's start with the Data Median Set, which is the Critical Point or the Optimum Value: $\Rightarrow X = -25$

b) The next greater value than the Critical Point will be the Mean of the Critical Point and the Greater x-intercept: $\Rightarrow \begin{cases} X_1 = 0 \\ X_2 = -50 \end{cases}$

- The Greater Intercept in this particular case is $X_1 = 0$ as $0 > -50$

$$\Rightarrow Mean = \frac{0 + (-25)}{2} = \frac{-25}{2} = -12.5$$

c) The next greater value of the Data Set will be the Greater x-intercept, $X_1 = 0$

$$\Rightarrow -25 < -12.5 < 0$$

d) The next Step is to determine the Increment Constant Value that will be the difference between the Greater x-intercept, $X_1 = 0$ and the Mean Value, -12.5

$$\Rightarrow \underline{\text{Increment Constant Value}} = 0 - (-12.5) = +12.5$$

e) The Next Greater Value of the Data Set will be to add the Greater x-intercept, $X_1 = 0$, to the Increment Constant Value, $+12.5$ $\Rightarrow 0 + 12.5 = +12.5$ and so on…

$$\Rightarrow \text{..........} -25 < -12.5 < 0 < +12.5 \text{..........}$$

f) <u>Keep in Mind</u>: The <u>Greater x-intercept</u> *must be between two values* V_1 *and* V_2 *that are Greater than the Critical Point*:

$$\Rightarrow [Critical - Point] < |V_1 < Greaterx - intercept < V_2| \Leftrightarrow [-25] < -12.5 < 0 < +12.5$$

g) Unless needed for more complex problem solving, usually <u>three values greater than the Critical Point and three values lesser than the Critical Point</u> should be more than enough to achieve a more accurate and precise Variation Table and Perfectly Symmetrical and Beautiful Parabola!

h) The <u>Next First Lesser Value</u> than the Critical Value of the Data Set will be determined by subtracting the <u>Increment Constant Value</u> from the <u>Optimum Value</u> or <u>the Critical Point</u>: $\Rightarrow -25 - (+12.5) = -37.5$

$$\Rightarrow -37.5 < -25 < 12.5 < 0 < +12.5$$

i) The Second Lesser Value than the Critical Point will be achieved by subtracting the Increment Constant Value from the First Lesser Value than the Critical Point: $\Rightarrow -37.5-(+12.5)=-50$

$$\Rightarrow \dots\dots\dots -50 < -37.5 < -25 < 12.5 < 0 < +12.5 \dots\dots\dots$$

j) The Third Lesser Value than the Critical Point will be achieved by subtracting the Increment Constant Value from the Second Lesser Value than the Critical Point.

$$\Rightarrow -50-(+12.5)=-62.5$$

$$\Rightarrow \dots\dots\dots -62.5 < -50 < -37.5 < -25 < 12.5 < 0 < +12.5 \dots\dots\dots$$

My eternal Obsession with a Perfect Symmetry obliges me to match the Colors with Respect to the Central Tendency, which is the Critical Point or the Optimum Value of the Data Set.

$$\Rightarrow \dots\dots\dots -62.5 < -50 < -37.5 < -25 < -12.5 < 0 < +12.5 \dots\dots\dots$$

k) Let's Justify the Surprising Constance of the Increment Value as follows:

$$\Rightarrow \begin{cases} 12.5-0=12.5 \\ 0-(-12.5)=12.5 \\ -12.5-(-25)=12.5 \\ -25-(-37.5)=12.5 \\ -37.5-(-50)=12.5 \\ -50-(-62.5)=12.5 \end{cases} \Rightarrow$$ **The Increment Value is indeed a constant of 12.5**

l) The Confidence Interval that fits perfectly our Variation Table is as follows:

$$\Rightarrow]-\infty .... -62.5 < -50 < -37.5 < -25 < -12.5 < 0 < +12.5 .... +\infty[$$

m) Limit of $P(X)$:

$$\Rightarrow \lim_{x \to \pm\infty} P(X) = \lim_{x \to \pm\infty} \left(10X^2 + 500X\right)$$

$$\Rightarrow \lim_{x \to \pm\infty} P(X) = \lim_{x \to \pm\infty} X^2\left(10 + \frac{500}{X}\right) \to +\infty$$

As $\lim_{x \to \pm\infty} \left(\frac{500}{X}\right) \to +0$

n) Input/Output Values:

- For $P(X) = 10X^2 + 500X \Rightarrow \frac{d}{dx}P(X) = 20X + 500$

- With respect to the following Confidence Interval:

$$\Rightarrow]-\infty..........-62.5 < -50 < -37.5 < -25 < -12.5 < 0 < +12.5..........+\infty[$$

$$\Rightarrow \begin{cases} P'(-62.5) = -750 \Rightarrow P(-62.5) = 7812.5 \\ P'(-50) = -500 \Rightarrow P(-50) = 0 \\ P'(-37.5) = -250 \Rightarrow P(-37.5) = -4687.5 \\ P'(-25) = 0 \Rightarrow P(-25) = -6250 \\ P'(-12.5) = +250 \Rightarrow P(-12.5) = -4687.5 \\ P'(0) = +500 \Rightarrow P(0) = 0 \\ P'(+12.5) = +750 \Rightarrow P(+12.5) = 7812.5 \end{cases}$$

- We notice a totally perfect symmetry. You call me crazy, but I really love it!

o) Summarized Table of Values:

x	P’ (x)	P (x)
–62.5	–750	7812.5
–50	–500	0
–37.5	–250	–4687.5
–25	0	–6250
–12.5	+250	–4687.5
0	+500	0
+12.5	+750	7812.5

p) Variation Table:

X	$-\infty$ ...	-62.5	-50	A = -37.5	-25	B = -12.5	0	$+12.5$	..... $+\infty$
$P'(x)=20X+500$	-	-750	-500	-250	0	$+250$	$+500$	$+750$	+
$P(x)=10x^2+500x$	$+\infty$	7812.5	0	-4687.5	-6250	-4687.5	0	$+7812.5$	$+\infty$
Parabola Opens Up (Behaior)	↘	↘	↘	↘	→	↗	↗	↗	↗
	Decreasing Function				Mini-mum $(-25,-6250)$	Increasing Function			

$P'(X) = 20X + 500$ $P(X) = 10X^2 + 500X$
Minimum point (vertex): $(-25, -6250)$

Conclusion:

The purpose in this First created Scenario was to figure out the exact values of X and Y that adhere to the two given constraints:

$$\Rightarrow \begin{cases} Y = 10X + 500 \langle 1 \rangle \\ Minimum : P(X) = X \times Y \langle 2 \rangle \end{cases}$$

* With respect to the Vertex, we have found through different methods:

$$\Rightarrow Vertex = V(-25, -6250)$$

* **The Minimum, -6250, was achieved when $X=-25$, which is obviously the equation that represents the Axis of Symmetry of this Beautifully Symmetrical Parabola.**

$$\Rightarrow \begin{cases} X \times Y = -6250 \\ X = -25 \end{cases} \Rightarrow -25 \times Y = -6250$$

$$\Rightarrow Y = \frac{-6250}{-25} = +250$$

* **The line $Y = 10X + 500 \langle 1 \rangle$ intersect with the Parabola $P(X) = 10X^2 + 500X$ at the following points:**

$$\Rightarrow 10X^2 + 500X = 10X + 500$$

$$\Rightarrow 10X^2 + (500X - 10X) - 500 = 0$$

$$\Rightarrow 10X^2 + 490X - 500 = 0$$

$$\Rightarrow X^2 + 49X - 50 = 0$$

$$\Rightarrow \begin{cases} X_1 = \dfrac{-49 - \sqrt{49^2 - 4(1)(-50)}}{2(1)} \\ X_2 = \dfrac{-49 - \sqrt{49^2 - 4(1)(-50)}}{2(1)} \end{cases}$$

$$\Rightarrow \begin{cases} X_1 = \dfrac{-49 - \sqrt{2401 + 200}}{2} = \dfrac{-49 - \sqrt{2601}}{2} = \dfrac{-49 - 51}{2} = \dfrac{-100}{2} = -50 \\ X_2 = \dfrac{-49 + \sqrt{2401 + 200}}{2} = \dfrac{-49 + \sqrt{2601}}{2} = \dfrac{-49 + 51}{2} = +\dfrac{2}{2} = +1 \end{cases}$$

* **The line and the Parabola intersect at the points:**

$$\Rightarrow \begin{cases} X_1 = -50 \\ X_2 = +1 \end{cases} \Rightarrow \begin{cases} Y_1 = 10X_1 + 500 \\ Y_2 = 10X_2 + 500 \end{cases} \Rightarrow \begin{cases} Y_1 = 10(-50) + 500 = 0 \\ Y_2 = 10(+1) + 500 = 510 \end{cases}$$

* **Consequently, the intersecting points are:**

$\Rightarrow (-50,0)$ **and** $(1,510)$

* **These intersecting points seem to satisfy the First Constraint, but certainly not the second.**

q) Verification:

For $X = -25$ and $Y = +250$, let's check together that the two conditions are indeed satisfied:

$$\Rightarrow \begin{cases} Y = 10X + 500 \langle 1 \rangle \\ X \times Y = -6250 \langle 2 \rangle \end{cases}$$

CONSTRAINTS	LEFT SIDE	RIGHT SIDE
$Y = 10X + 500 \langle 1 \rangle$	$Y = +250$	$10(-25) + 500 = +250$
$X \times Y = -6250 \langle 2 \rangle$	$X \times Y = (-25) \times (+250) = -6250$	-6250

* **We see that both sides of the equation are verified for both constraints:**

$$\begin{cases} Y = 10X + 500 \langle 1 \rangle \\ X \times Y = -6250 \langle 2 \rangle \end{cases}$$

* **Consequently, the two Numbers that satisfy both conditions $\langle 1 \rangle$ and $\langle 2 \rangle$ are:**

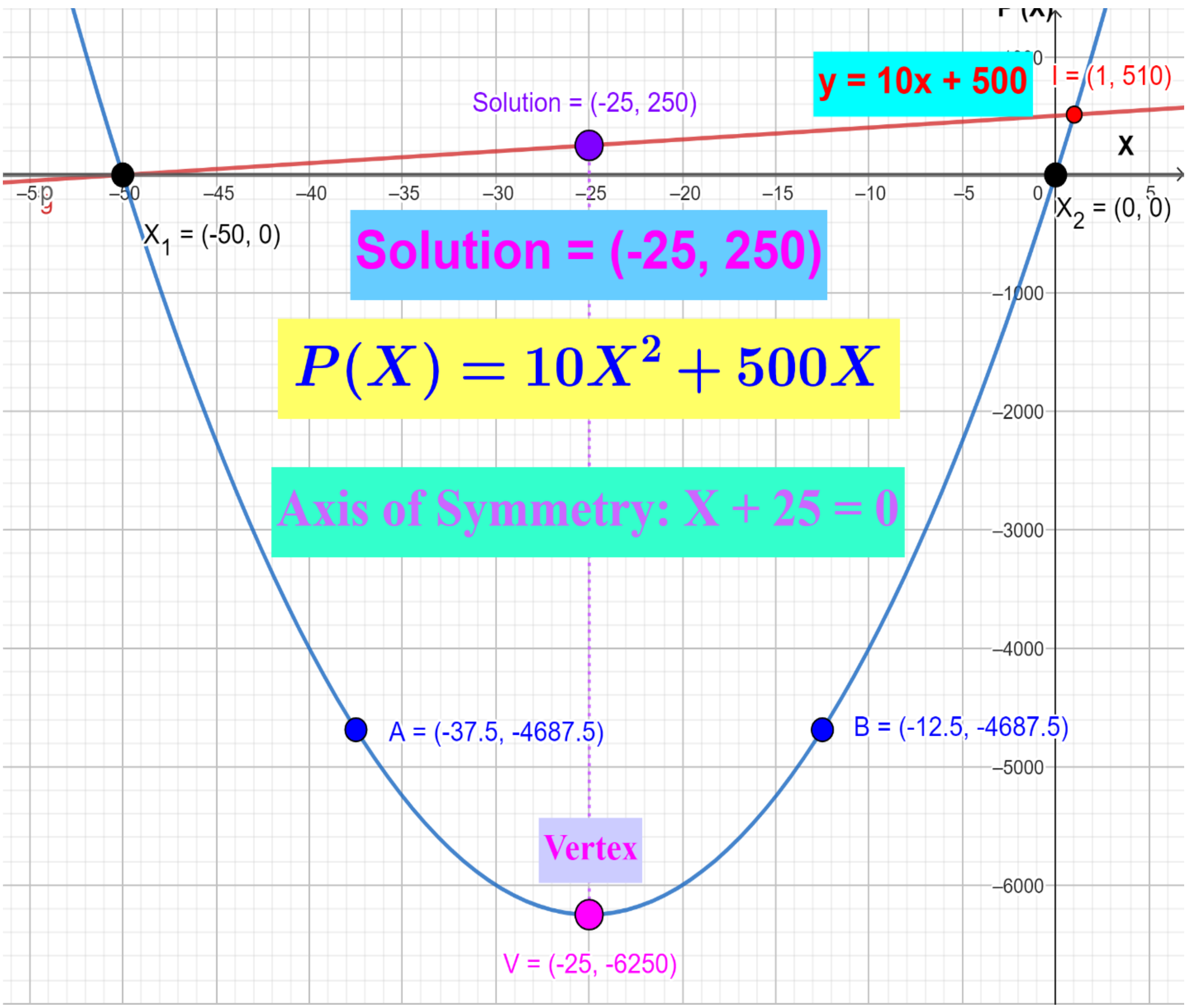
$$\Rightarrow \begin{cases} X = -25 \\ Y = +250 \end{cases}$$

r) Graphical Representation of this First Created Scenario:

* **We have all the necessary data and information that help us achieve the most accurate, beautiful, and perfectly symmetrical Parabola that thoroughly explains the Current Created Case Scenario that we have created together.**

* **Keep in mind one thing: Your careful attention and total understanding of the Concept of Quadratic Functions, Equations, Inequalities, and Related Engineering Applications will contribute, in part, to the most Worldwide Positive Outcome: Unconditional Love for Mathematics, the only tangible and unbiased Universe of Truth that helps us remain united, as Human Beings, achieve Divine Intelligence and, most of all, explore the Beauty and the immensity of our Infinite, Magnificent Multidimensional Universe that we seem to be taking for granted. Godspeed.**

Graph 27: $P(X) = 10X^2 + 500X \langle 1 \rangle$ Versus $Y = 10X + 500 \langle 2 \rangle$



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